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МАТЕМАТИЧЕСКИЕ МЕТОДЫ И ИНФОРМАЦИОННЫЕ ТЕХНОЛОГИИ В ХИМИИ И ХИМИЧЕСКОЙ ТЕХНОЛОГИИ

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THE LOGISTIC MAP AND THE BIRTH OF PERIOD-3 CYCLE

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The goal of this paper is to present a proof that for the logistic map $x_{n+1} = f(x_n) = bx_n(1-x_n)$ the period-3 begins at $1+2\sqrt{2}$. The third-iterate map $f^3(x)$ is the key for understanding the birth of the period-3 cycle. Any point x in a period-3 cycle repeats every three iterates by definition. Such points satisfy the condition $x = f^3(x)$, and they are therefore fixed points of the third-iterate map. This fact and the so called tangent bifurcation for the logistic map, as well as the fixed points definition, are used for finding the $1+2\sqrt{2}$ value. The algebraic treatment utilizes some properties of symmetric polynomials in three variables. For the purposes of this paper, the bifurcation diagram for the logistic map is also presented, as well as a program in Mathematica for its construction.

Key words: *attractor, bifurcation diagram, dynamical systems, chaos, fixed points, logistic map, maps, symmetric polynomials, tangent condition, vector fields.*

Introduction

Chaos and Dynamical Systems

At the beginning of the 20th century, Henry Poincaré (1854-1912) discovered the possibility of chaotic motion in celestial mechanics problems. When studying the model of a bisolar system with just a planet, Poincaré indicated that system can developed motion with random characteristics, in spite of, it is governed by strict laws of newtonionan mechanics.

In the biology field, are found species which population in a territory (birds in a forest, a bacteria that infest a human) tend to a normal level; in others in which the population varies each certain period (shortage and abundance time), and finally others, in which the number of individuals vary without a regular pattern.

The Fluids present regular motion, in which the particles follow streamlines that separate slowly in a linear way in time. Besides, two particles that pass by the same point with a small interval in time follow the same streamline. However, either previous characteristics appear in what called turbulent flow, in which the particles that were near each other in a certain moment are quickly separated, in a exponential way in time. Their trajectories do not keep any relation in a short time, because of that it is said that they forget the initial conditions.

The cited examples, belonging in appearance to different disciplines, remark an important fact: although being deterministic, present an unpredictable behavior. The exemplified systems are studied by the *Dynamical System Theory* where the *chaos* emerges. A dynamical system involves two components: (1) The notion of *state*, i.e., essential information about the system; (2) A *dynamics*, i.e., a rule that describes the evolution of the system state in time. The deterministic classical dynamical systems are grouped in two categories: (a) the *discrete*, in which one or more variables take values in discrete periods of time. From a mathematical point of view, they are called *maps*. A typical example of this situation could be the number of individuals of a biological specie in a specific territory; (b) the *continuous*, in which the variables are functions of time that obey to differential equations, such a pendulum subject to a force that varies periodically in time. From a mathematical point of view, they are called *vector fields*.

One of the most surprising results of physics in the last 40 years, is the verification that most of classical deterministic dynamical system present complex motions when time increases, and makes them unpredictable about their final state. These kind of system are called turbulents, chaotics and stochastics. The main characteristic of a chaotic system is its sensibility to small variation in the initial conditions. Starting from two proximate states and letting the system evolves in time, after certain period of time the two trajectories followed by system do not look like each other. It is said that the system forgets the initial conditions or that it does not has memory of the past. In others words, in a chaotic system a small change in the present, causes bigger change in the future.

The regular and chaotic behavior of dynamical systems reveals a probabilistic vision of the world, in which causal deterministic chains are linked and end when all the information about their initial state is completely destroyed. In this way, order and chaos, determinism and probabilities get together and complement each other, having as result a *complex world* full of possibilities than the cold mechanist vision.

1. The logistic map

The *logistic map* is a formula for approximating the evolution of an animal population over time. Many animal species are fertile only for a brief period during the year and the young are born in a particular season so that by the time they are ready to eat solid food it will be plentiful. For this reason, the system might be better described by a discrete difference equation that a continuous differential equation. Since not every existing animal will reproduce (a portion of them are male after all), not every female will be fertile, not every conception will be successful, and not every pregnancy will be successfully carried out to term; the population increase will be some fraction of the present population. Therefore, if x_n is the number of animals this year and x_{n+1} is the number the next year, then

$$x_{n+1} = bx_n, \quad (1)$$

where b is the *growth rate* or *fecundity*, will approximate the evolution of the population. However, this model produces an exponential growth without limit. Since every population is bound by the physical limitations of its surrounding, some allowance must be made to restrict this growth. If there is a carrying-capacity of the environment then the population may not exceed that capacity. If it does, the population would become extinct. This can be modeled by multiplying the population by a number that approaches zero as the population approaches its limit. If the x_n is normalized to this capacity, then the multiplier $(1-x_n)$ will suffice and the resulting logistic map becomes

$$x_{n+1} = bx_n(1-x_n). \quad (2)$$

For analyzing the map (2), the restrictions $0 < b \le 4$, $x \in [0,1]$ will be used for that map, maps the interval x into itself.

As it was said, the main characteristic of a chaotic system is its sensibility to small variation in the initial conditions. Starting from two proximate states and letting the system evolves in time, after certain period of time the two trajectories followed by system do not look like each other. This is exemplified for the logistic map in Figure 1, where the trajectories corresponding to seeds $x_0 = 0.2$, and $x_0 = 0.205$ for $b = 3.8$, are quite different, when n increases.

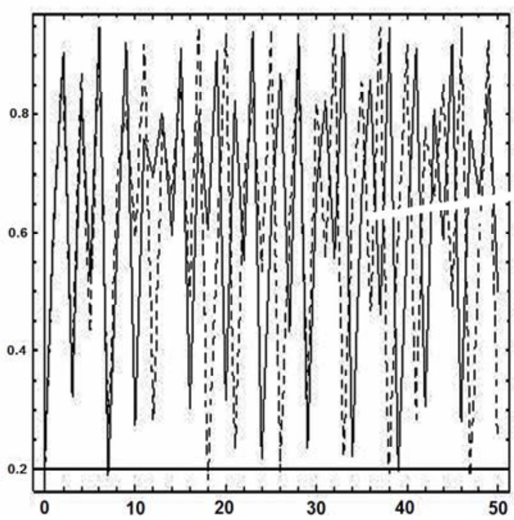

Fig. 1. Two solutions of $x_{n+1} = bx_n(1-x_n)$, $b = 3.8$, $x_0 = 0.2$, $x_0 = 0.205$.

The long-term behavior for *all* values of b at once, is shown in the Figure 2, known as *bifurcation diagram*, a magnificent picture that has become an icon of nonlinear dynamics. The Figure 2 plots the system's attractor as a function of b for $3 \le b \le 4$. To generate the bifurcation diagram, it is necessary to write a computer program with two loops. First, choose a value of b. Then generate an orbit starting from some random initial condition x_0. Iterate for 1000 cycles or so, to allow the system settle down its eventual behavior. Once the transient have decayed, plot many points, say $x_{1001}, \ldots, x_{1500}$ above that b. Then move to an adjacent value of b and repeat, eventually sweeping across the whole picture. The author used the following Mathematica program for drawing the bifurcation diagram. The program allows the reader to change the b interval.

```
f[b_, x_]:=b x (1-x)
IterMap[b_, x_,n_]:=Module[{x0,y0,li},x0=x;li={{0,x0}};For[i=1,i<=n,i++,y0=f[b,x0];x0=y0;
li=Append[li,{i,x0}]];li]
Bifurcation[li_List,x0_,n1_,n2_]:=Module[{r,graf,m,i,li1,li2},m=Length[li];graf={};
Do[r=li[[i]];li1=NestList[IterMap,{r,x0},n1+n2];li2=Take[li1,{n2,n1}];graf=Append[graf,li2],{i,m}];
ListPlot[Flatten[graf,1],AxesLabel->{"b", "x"}]];
li=Table[3.4+ .00160 i,{i,350}];Bifurcation[li,.2,600,300];
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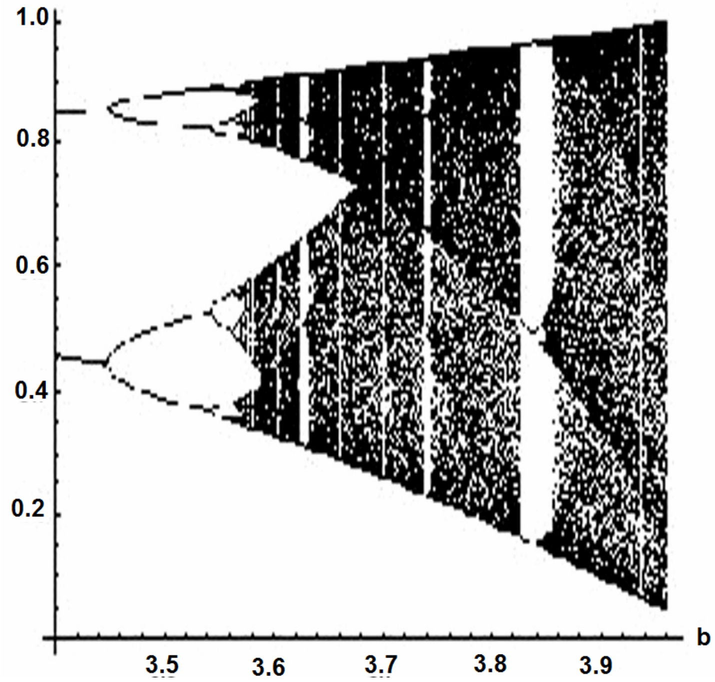

Fig. 2. Bifurcation diagram for $x_{n+1} = bx_n(1 - x_n)$.

The Figure 2 shows the most important part of the diagram, in the region $3.4 \le b \le 4$. At b=3.4, the attractor is a period-2 cycle, as indicated by the two branches. As b increases, both branches split simultaneously, yielding a period-4 cycle. A cascade of further period-doublings occurs as b increases, yielding period-8, period-16, and so on, until at $b = b_\infty \approx 3.57$, the map becomes chaotic and the attractor changes from a finite to infinite set of points.

For $b > b_\infty$ the bifurcation diagram reveals an unexpected mixture of order and chaos, with *periodic windows* interspersed between chaotic clouds of points. The large window beginning near $b \approx 3.83$ contains a stable period-3 cycle.

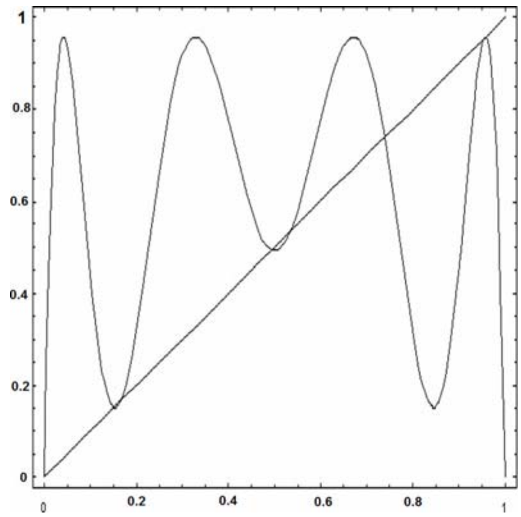

Fig. 3. Graph of $f^3(x)$ versus x for $b = 3.835$.

2. Algebraic treatment of period-3 cycle

The key for obtaining that the period-3 begins at $b = 1 + 2\sqrt{2}$, is the third-iterate map $f^3(x)$. Any point x in a period-3 cycle repeats every three iterates, by definition, so such points satisfy $x = f^3(x)$ and are fixed points of the third-iterate map. These fixed points cannot be obtained explicitly because $f^3(x)$ is an eight-degree polynomial. However a graph provides sufficient insight. The intersection between the graph and the diagonal line correspond to solutions of $x = f^3(x)$, as shown in Figure 3.

Now decreases b toward chaotic regime. Then the graph in Figure 3 changes its shape: The hills move down and the valleys rise up. The Figure 4 shows that when $\mathrm{b} = 3.8$, the intersections with the diagonal have vanished. Hence, for some intermediate value between $\mathrm{b} = 3.8$ and $b = 3.835$, the graph of $f^3(x)$ must have become *tangent* to the diagonal. At this critical value of b, the stable and unstable peridod-3 cycles coalesce and annihilate in a *tangent bifurcation*. This transition defines the birth of the period-3 cycle.

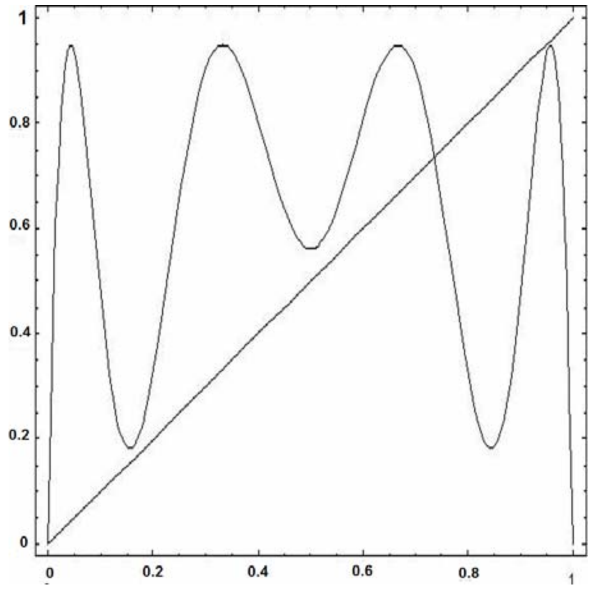

Fig. 4. Graph of $f^3(x)$ versus x for $b = 3.8$.

Proposition 1 (The Birth of Period-3 Cycle). *In the logistic map, the period-3 begins at* $1 + 2\sqrt{2}$

Proof: Let x_1, x_2, x_3 be the fixed points of $f^3(x)$. The tangent condition means that the derivative of $f^3(x)$ at the fixed points must be 1. By chain rule, this derivative is the product $f'(x_1)f'(x_2)f'(x_3)$.

Since $f'(x) = b(1-2x)$, then $b(1-2x_1)(1-2x_2)(1-2x_3) = 1$. After performing operations, the following result follows:

$$b^3(1 - 2(x_1 + x_2 + x_3) + 4(x_1x_2 + x_1x_3 + x_2x_3) - 8x_1x_2x_3) = 1. \tag{3}$$

Let $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1x_2 + x_1x_3 + x_2x_3$ and $s_3 = x_1x_2x_3$. The expressions s_1, s_2, s_3 are called *fundamental symmetric polynomials*. With these variables change, the tangent condition becomes:

$$b^3(1 - 2s_1 + 4s_2 - 8s_3) - 1 = 0. \tag{4}$$

From fixed points definition, the following equations are obtained:

$$b(x_1 - x_1^2) = x_2, b(x_2 - x_2^2) = x_3, \ b(x_3 - x_3^2) = x_1. \tag{5}$$

By summing equations (5), it is obtained

$$b((x_1 + x_2 + x_3) - (x_1^2 + x_2^2 + x_3^2)) = x_1 + x_2 + x_3. \tag{6}$$

Now

$$(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_1x_3 + x_2x_3). \tag{7}$$

This means that

$$x_1^2 + x_2^2 + x_3^2 = s_1^2 - 2s_2. \tag{8}$$

Then, the Equation (6) can be written as

$$b(s_1 - s_1^2 + 2s_2) = s_1. \tag{9}$$

Solving for s_2

$$s_2 = \frac{s_1(1-b) + bs_1^2}{2b}. \tag{10}$$

By multiplying equations (5), the following equation becomes

$$b^3(x_1 - x_1^2)(x_2 - x_2^2)(x_3 - x_3^2) = x_1x_2x_3 \tag{11}$$

The result of these operations is

$$\begin{aligned} &b^3(x_1x_2x_3 - x_1x_2x_3^2 - x_1x_2^2x_3 + x_1x_2^2x_3^2 - x_1^2x_2x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_3 - (x_1x_2x_3)^2) \\ &= b^3(x_1x_2x_3 - x_1x_2x_3(x_1 + x_2 + x_3) + x_1x_2x_3(x_1x_2 + x_1x_3 + x_2x_3) - (x_1x_2x_3)^2) = x_1x_2x_3. \end{aligned} \tag{12}$$

In new variables, after cancelling common factor $x_1x_2x_3$, the Equation (12) is

$$b^3(1-s_1+s_2-s_3)=1. \qquad (13)$$

Solving for s_3

$$s_3=\frac{b^3(1+s_2)-b^3s_1-1}{b^3}. \qquad (14)$$

Now, by multiplying the first equation of equations (5) by x_1, the second by x_2 and the third by x_3 and summing, it is obtained

$$b((x_1^2+x_2^2+x_3^2)-(x_1^3+x_2^3+x_3^3))=x_1x_2+x_1x_3+x_2x_3. \qquad (15)$$

It is easy to show that

$$x_1^3+x_2^3+x_3^3=(x_1+x_2+x_3)^3-3(x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)+3x_1x_2x_3= \\ =s_1^3-3s_1s_2+3s_3. \qquad (16)$$

By plugging (8) and (16) into (15), this equation becomes

$$b(s_1^2-2s_2-s_1^3+3s_1s_2-3s_3)-s_2=0. \qquad (17)$$

The Equation (17) defines a relation between s_1 and b, because s_2 and s_3 are functions of b and s_1. Now this relation will be obtained and analyzed.

By replacing equations (10) and (14) into Equation (17), and after simplifications, this equation follows

$$\frac{b}{2}s_1^3+(1-3b)s_1^2+(-2-\frac{1}{2b}+\frac{11b}{2})s_1+(\frac{3}{b^2}-3b)=0. \qquad (18)$$

This equation can be factored as follows

$$\frac{(3-3b+bs_1)(2+2b+2b^2-(b+3b^2)s_1+b^2s_1^2)}{2b^2}=0. \qquad (19)$$

The roots of Equation (19) are

$$s_{11}=3-\frac{3}{b},\ s_{12}=\frac{b+3b^2+b\sqrt{b^2-2b-7}}{2b^2},\quad s_{13}=\frac{b+3b^2-b\sqrt{b^2-2b-7}}{2b^2}. \qquad (20)$$

Now, the roots s_{11}, s_{12} and s_{13} will be analyzed by using Equation (3) (tangent condition). As a function of s_1 and b, this equation is

$$2b^3s_1^2+(2b^2-8b^3)s_1+7b^3-7=0. \qquad (21)$$

By replacing s_{11} into (21), it is obtained

$$b^3-6b^2-12b+7=(b-1)(b^2-5b+7)=0. \qquad (22)$$

The roots of this equation are

$$b_1=1,\quad b_2=(5+i\sqrt{3})/2,\quad b_3=(5-i\sqrt{3})/2. \qquad (23)$$

All of these roots must be discarded because there are two imaginary roots and the real root $\mathrm{b}_1=1<3.8$.

Since s_1 must be real, s_{12} and s_{13} must be also real (see equations (20)). This means that

$$b^2-2b-7=0. \qquad (24)$$

The roots of this equation are

$$b_{11}=1+2\sqrt{2},\quad b_{12}=1-2\sqrt{2}. \qquad (25)$$

Then, $s_1=s_{12}=s_{13}$, i.e.,

$$s_1=\frac{b+3b^2}{2b^2}=\frac{3b+1}{2b}. \qquad (26)$$

Since $b_{12}=1-2\sqrt{2}<3.8$, this root must be discarded. By replacing (26) into first member of Equation (21), it is obtained $3b+4b^2-b^3-14$, and plugging $b=1+2\sqrt{2}$ into this expression, it follows that

$$3(1+2\sqrt{2})+4(1+2\sqrt{2})^2-(1+2\sqrt{2})^3-14=3+6\sqrt{2}+4+16\sqrt{2}+32-1-6\sqrt{2}-24-16\sqrt{2}-14$$
$$=39\text{-}25\text{-}14=0$$

This means that the tangent bifurcation is satisfied by $b=1+2\sqrt{2}$. Therefore, the b value at which the period-3 cycle begins is $b=1+2\sqrt{2}$. Now, the proof of **Proposition 1** is complete.

With this value of b, now it is possible to find the s_1, s_2, s_3 values from equations (10), (14) and (26). These values are:

s_1=1.63060193748187 , s_2=0.7270901428157046, s_3=0.07866692728074915

From definition of s_1, s_2, s_3, it is possible to find x_1, x_2, x_3 by solving the following equations system:

$$\begin{aligned} &x_1+x_2+x_3=1.63060193748187\\ &x_1x_2+x_1x_3+x_2x_3=0.7270901428157046 \qquad (27)\\ &x_1x_2x_3=0.07866692728074915 \end{aligned}$$

For solving the system (27), the author used the following Mathematica program that implements de Newton's method.

```
x[x1_,x2_,x3_,n_]:=Module[{f1,f2,f3,i,m,s1,s2,s3,u,v,x01,x02,x03},
s1=1.63060193748187; s2=0.7270901428157046; s3=0.07866692728074915;
x01=x1; x02=x2; x03= x3;
For[i=1,i <=n,i++,f1=x01+x02+x03-s1;f2=x01 x02+x01 x03+x02 x03-s2;
f3=x01 x02 x03-s3; m={{1,1,1},{x02+ x03, x01 +x03,x01+ x02},
{x02 x03,x01 x03, x01 x02}}; v={f1,f2,f3};
u=LinearSolve[m,v]; y1=x01-u[[1]];y2=x02-u[[2]];
y3=x03-u[[3]]; x01=y1; x02=y2; x03=y3]; {x01,x02,x03}];
x[.5, .9,.15,15]
```

The result is:

$x_1=0.5143552770619904, x_2=$ $0.9563178419736228, x_3=0.1599288184462569$.

By replacing these values into equations (5) it is easy to show that they satisfy those equations, and they represent the x values at which the period-3 cycle, in the logistic map, begins.

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