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THEORETICAL BASES OF CHEMICAL TECHNOLOGY

ANALYTIC SOLUTION OF NONLINEAR LEYBENSON EQUATION IN THE THEORY OF FILTRATION

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Analytic solution of nonlinear Leybenson equation in the theory of filtration is obtained. Analytical solutions of the partial differential equations are presented in the explicit algebraic form. The integral surfaces in three dimensions are presented.

Keywords: *Leybenson equation, theory of filtration.*

1. Introduction

The theory of filtration is a branch of hydrodynamics studying the motion of liquids through porous media, that is, through media penetrated by a system of intercommunicating void spaces. It is the practice to consider the motion of a liquid upon filtration as a certain effective continuous flow. So, filtrational flows formally have similarity to flows in pipes and channels, and terms of the theory of filtration coincide in many respects with the hydraulic terminology. Nevertheless, let us recall some definitions in the theory of filtration.

The mentioned porous media are called filtering or permeable. Some grounds (sands, sand clays and clay loams), construction materials (crushed stone, porous concrete and brickworks) can be examples of filtering media. The permeability of a porous medium is determined experimentally. Aquiclude is ground almost not passing water. Clays are often waterproof, because their pores are closed and small. As for an impenetrable construction material, it is usually called damp-proof instead of water-proof. The theory of filtration in the context of construction, water supply and wastewater disposal considers regularities of water filtration for carrying out quantitative calculations, for example, when designing drainage systems (drainages) lowering the level of ground waters in order to protect underground constructions and premises of buildings against flooding. A special role in ecology is played by the motion of moisture in the soil. The correct organization of irrigation in the motion of soil moisture is one of the most important tasks of the theory of filtration. Methods of the theory of filtration are used when solving the problem of ground waters protection from pollution by production wastes, fertilizers and other waste products of mankind. The main energy sources of the 20th century – oil and gas – are produced from deeply lying underground layers. Accumulation of oil and gas in these porous collector layers and the main technologies of extracting (producing) are governed by the laws of the theory of filtration.

Porosity ε is the most important quantitative characteristic of porous bodies. It is defined as the volume fraction of a body occupied by pores, or the volume of pores in the volume unit of a material. This definition usually ignores closed isolated pores: it considers only interconnected open-ended pores. They form the pore space, a complex branched and irregular network of pores. The porosity of most materials ranges within 0.1–0.4. The ability of a porous medium to pass a liquid is characterized by permeability. Its definition is closely related to the fundamental law of the motion of a liquid in a porous medium called Darcy's law in honor of the French engineer Henri Darcy, who experimentally established this law in 1856. In continuum mechanics, when studying the flow of liquids and gases in a porous medium in a gravitational field, the differential form of Darcy's law is applied:

$$\mathbf{u} = -\frac{k}{\mu}\operatorname{grad}(\rho g z + p) , \qquad (1.1)$$

where p is external pressure; ρ is the fluid density; μ is its dynamic viscosity; g is acceleration of gravity; z is a vertical coordinate. In the equation (1.1) k is a proportionality coefficient, which is a characteristic of the porous environment and does not depend on the sample size and the liquid properties. Darcy's law is true at a slow flow of a liquid, i.e., at small Reynolds numbers. The theory of ground water motion deals only with water, the viscosity of which $\mu = 10^{-3}$ Pa·s, and density $\rho = 10^3$ kg/m^3. The classical theory of filtration discussed up to now deals with the flow of a uniform liquid in a porous medium. In most modern applications, however, it is necessary to consider non-uniform systems, multicomponent multiphase mixes. Let's mention only such applications of the theory of filtration as pollution of ground waters, migration of moisture in a soil layer and replacement of oil by stratal or artificially pumped water or gas (see [1–5]).

The foundations of the theory of gas flow in a porous medium were developed by the founder of the Soviet school of oil-and-gas hydromechanics L.S. Leybenson. He derived for the first time differential equations of non-stationary filtration of a perfect gas in a layer according to Darcy's law. The nonlinear differential equation derived by him was later referred to as Leybenson equation. Unlike hyperbolic Boussinesq equation, Leybenson equation is of the parabolic type. When deriving the latter equation, it was assumed that the porosity and permeability factors do not change with changing pressure (the layer is not deformed), the gas viscosity does not depend on pressure either, and the gas is perfect. It is usually assumed also that the gas filtration in the layer is isothermal, i.e., the gas and layer temperature remains constant.

The basic variant of one-dimensional non-stationary theory of filtration gives the following non-linear differential equation:

$$\frac{\partial h}{\partial t} = \frac{\rho g k}{2\mu\varepsilon}\frac{\partial^2}{\partial x^2}h^2, \tag{1.2}$$

Dependent variable h is usually called static head. In the Russian literature equation (1.2) is known also as a special case of Leybenson equation. Equation (1.2) appears, for example, in the theory of non-stationary one-dimensional filtration of soil water adjoining to some rectangular reservoir having the height of water level $h = h_{max}$ at start time [3]. The permeable lateral surface of the reservoir is a source of the motion of the water spreading along axis x (see Fig. 1). Let's present briefly the derivation of Leybenson equation, which will allow finding out at the same time the assumptions on which it was obtained.

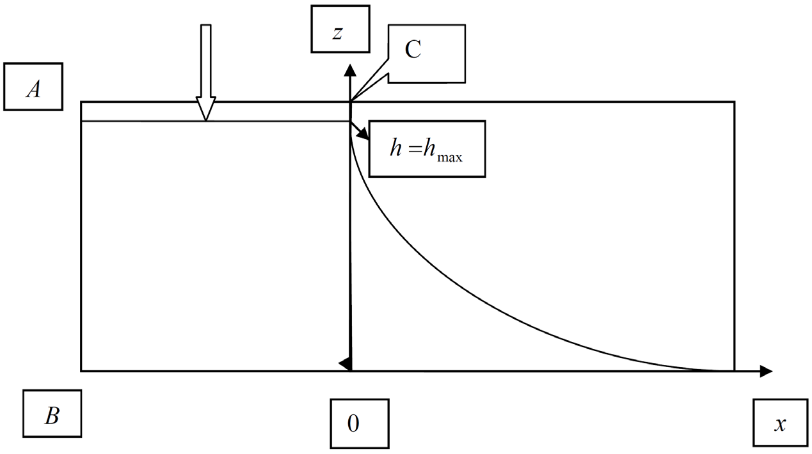

Fig. 1. Illustration to the derivation of the one-dimensional non-stationary Leybenson equation.

Let us assume that the liquid contained in AB0C reservoir infiltrates into the soil in the direction $0x$. The liquid motion is considered to be one-dimensional. For this purpose it is assumed that elements AB and B0x are impenetrable for the liquid. It means, using the conventional terminology, that a one-dimensional non-stationary ground free-flow with a horizontal aquiclude is considered. A similar assumption is widely used in the theory of radial gas flow to a well bore.

Neglecting the influence of the inertia terms upon the motion of the liquid along axis z we have

$$-\frac{\partial p}{\partial z} - \rho g = 0. \tag{1.3}$$

From (1.3) it follows that $p + \rho g z = const$, and if $z = 0$, then $p = \rho g h$. That is,

$$p + \rho g z = \rho g h, \tag{1.4}$$

and Darcy's law can be written as follows:

$$\mathbf{u} = -\frac{k}{\mu}\operatorname{grad}(\rho g h). \tag{1.5}$$

Les us introduce control volume $(x, x+dx)\times 1\times(0,h)$ as a vertical rectangular parallelepiped with height h and base area $(1\times dx)$. For the direction x we have velocity $u_x = -\frac{k}{\mu}\rho g\frac{\partial h}{\partial x}$, and the flow of matter through the lateral surface of the mentioned parallelepiped is

$$q_x = \rho u_x h = -\frac{k}{\mu}\rho g h\frac{\partial h}{\partial x} = -\frac{kg}{2\mu}\rho^2\frac{\partial}{\partial x}h^2. \tag{1.6}$$

The mass of liquid q_x comes to the parallelepiped in a unit time, and the mass $q_x + \frac{\partial q_x}{\partial x}dx$ flows from it.

For the control volume containing an incompressible liquid we can write the mass balance per time unit.

$$\varepsilon \frac{\partial h}{\partial t}\rho = \frac{\partial q_x}{\partial x} , \qquad (1.7)$$

or

$$\frac{\partial h}{\partial t} = -\frac{kg\rho}{2\mu\varepsilon}\frac{\partial^2}{\partial x^2}h^2 . \qquad (1.8)$$

Equation (1.8) is the nonlinear one-dimensional non-stationary Leybenson equation written above in the form (1.2).

2. Leybenson equation and its analytical solution

The basic variant of one-dimensional non-stationary theory of filtration gives the following nonlinear differential equation (see (1.2)):

$$\frac{\partial h}{\partial t} = \frac{\rho g k}{2\mu\varepsilon}\frac{\partial^2}{\partial x^2}h^2 , \qquad (2.1)$$

or

$$\frac{\partial h}{\partial t} = \frac{\rho g k}{\mu\varepsilon}\left[h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2\right] . \qquad (2.2)$$

Let us reduce equation (2.2) to a dimensionless form using $h = h_{max}$ as a scale, that is, $\tilde{h} = h / h_{\max}$, $\tilde{x} = x / h_{\max}$. Then

$$\tilde{t} = t\frac{\rho g k}{\mu\varepsilon h_{\max}} , \qquad (2.3)$$

and equation (2.2) is reduced to the following dimensionless form:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} = \tilde{h}\frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} + \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right)^2 . \qquad (2.4)$$

In order to simplify the symbols, we shall omit the tilde sign in further transformations of equation (2.4). Let us seek for the solution of (2.4) in the form of a tripartite polynom with coefficients depending only on time. We have

$$h(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t). \qquad (2.5)$$

In this case

$$\frac{\partial h}{\partial t} = \frac{\partial \varphi}{\partial t}x^2 + \frac{\partial \psi}{\partial t}x + \frac{\partial \chi}{\partial t} , \qquad (2.6)$$

$$\frac{\partial h}{\partial x} = 2x\varphi(t) + \psi(t) , \qquad (2.7)$$

$$\frac{\partial^2 h}{\partial x^2} = 2\varphi(t). \qquad (2.8)$$

Substitution with (2.6) – (2.8) in equation (2.5) gives

$$\frac{\partial \varphi}{\partial t}x^2 + \frac{\partial \psi}{\partial t}x + \frac{\partial \chi}{\partial t} = 2\varphi(t)\left(\varphi(t)x^2 + \psi(t)x + \chi(t)\right) + \left(2x\varphi(t) + \psi(t)\right)^2 \qquad (2.9)$$

or

$$\frac{\partial \varphi}{\partial t}x^2 + \frac{\partial \psi}{\partial t}x + \frac{\partial \chi}{\partial t} = 6\varphi^2(t)x^2 + 6\varphi(t)\psi(t)x + 2\varphi(t)\chi(t) + \psi^2(t). \qquad (2.10)$$

Equation (2.10) is an identity that is true at all values of . This is possible only under the following conditions:

$$\frac{\partial \varphi}{\partial t} = 6\varphi^2 , \qquad (2.11)$$

$$\frac{\partial \psi}{\partial t} = 6\varphi\psi , \qquad (2.12)$$

$$\frac{\partial \chi}{\partial t} = 2\varphi\chi + \psi^2 . \qquad (2.13)$$

Now let us solve the set of equations (2.11) – (2.13). Equation (2.11) is Bernoulli equation, which is easily integrated by substituting with $w = 1/\varphi$:

$$w' = -6$$

or

$$\varphi = -\frac{1}{6t + C} . \qquad (2.14)$$

Equation (2.12) can be written now as

$$\frac{\partial \psi}{\partial t} = -6\frac{1}{6t + C}\psi \qquad (2.15)$$

or

$$d\ln\psi = -\frac{1}{6t + C}d(6t + C), \qquad (2.16)$$

which gives the following solution:

$$\psi = \frac{B}{6t + C} ; \qquad (2.17)$$

B and C are integration constants. Let us transform equation (2.13) using solutions (2.14) and (2.17). We have

$$\frac{\partial \chi}{\partial t} + 2\frac{1}{6t+C}\chi = \frac{B^2}{(6t+C)^2}\ . \qquad (2.18)$$

The solution of linear non-uniform differential equation (2.18) with variable coefficients is known and given by

$$\chi = Ae^{-F} + e^{-F}B^2\int e^F \frac{1}{(6t+C)^2}dt\ , \qquad (2.19)$$

where

$$F(t) = 2\int \frac{1}{6t+C}dt\ , \qquad (2.20)$$

or

$$F(t) = \frac{1}{3}\ln(6t+C)\ . \qquad (2.21)$$

Using (2.19) and (2.21) gives

$$F(t) = \frac{1}{3}\ln(6t+C) \qquad (2.22)$$

or, after integration,

$$\chi = A(6t+C)^{-\frac{1}{3}} - \frac{1}{4}B^2(6t+C)^{-1}\ . \qquad (2.23)$$

As a result of solving three ordinary differential equations (2.11) – (2.13) we obtained three integration constants, which are included into solution (2.5):

$$h(x,t) = -\frac{1}{6t+C}x^2 + \frac{B}{6t+C}x + A(6t+C)^{-\frac{1}{3}} - \frac{1}{4}B^2(6t+C)^{-1}\ . \qquad (2.24)$$

which can be written in the following form:

$$h(x,t) = \frac{A}{(6t+C)^{1/3}} - \frac{1}{6t+C}\left[x^2 - x + \frac{1}{4}B^2\right]. \qquad (2.25)$$

Note that at start time

$$h(x,0) = h(x) = \frac{A}{C^{1/3}} - \frac{1}{C}\left[x^2 - x + \frac{1}{4}B^2\right], \qquad (2.26)$$

and for the origin

$$h(0,t) = \frac{A}{(6t+C)^{1/3}} - \frac{1}{4}\frac{B^2}{6t+C}\ . \qquad (2.27)$$

Let us assume that   by the data. Then

$$\mathrm{A} = C^{1/3}\left[1 + \frac{B^2}{4C}\right] \qquad (2.28)$$

and equation (2.25) takes the form

$$h(x,t) = \left[1 + \frac{B^2}{4C}\right]\frac{C^{1/3}}{(6t+C)^{1/3}} - \frac{1}{6t+C}\left[x^2 - x + \frac{1}{4}B\ . \qquad (2.29)\right.$$

In case of

$$h(x,t) = C^{1/3}\left[1 + \frac{1}{4C}\right]\frac{1}{(6t+C)^{1/3}} - \frac{1}{6t+C}\left(x - \frac{1}{2}\right)^2\ . \qquad (2.30)$$

Constants B and C are determined by the specific flow conditions, kinetic coefficients and properties of the porous medium. Let us give the calculations carried out for various C and B combinations as examples (Fig. 2 – 8). The determinative dimensionless parameters C and B in the presented results of mathematical modeling change by several orders of magnitude. The integral surfaces are constructed by means of the applied program pack Maple. The following, designations are used: $\mathrm{y(x,t)} \leftrightarrow \tilde{h}(\tilde{x},\tilde{t})$, $\mathrm{x} \leftrightarrow \tilde{x}$, $\mathrm{t} \leftrightarrow \tilde{t}$.

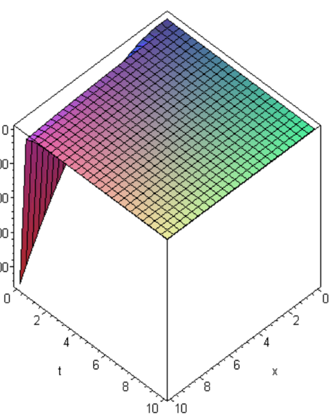

Fig. 2. Integral surface $\mathrm{y(x,t)}$ in case of $C = 0.1$, $B = 1$.

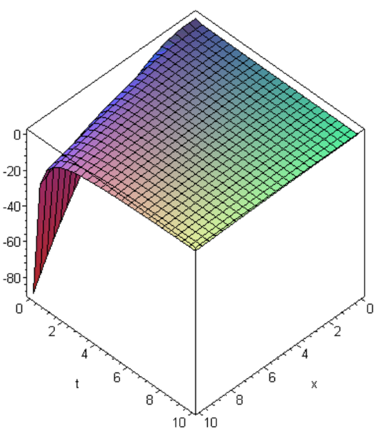

Fig. 3. Integral surface $\mathrm{y(x,t)}$ in case of $C = 1$, $B = 1$**.**

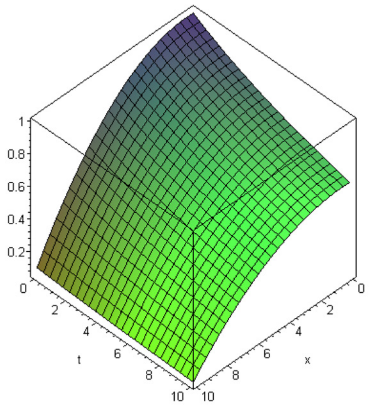

Fig. 4. Integral surface $y(x,t)$ in case of $C = 100$, $B = 1$.

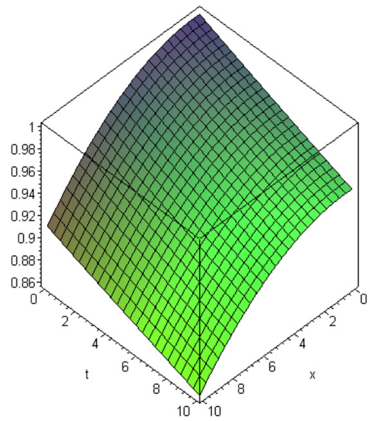

Fig. 5. Integral surface $y(x,t)$ in case of $C = 1000$, $B = 1$**.**

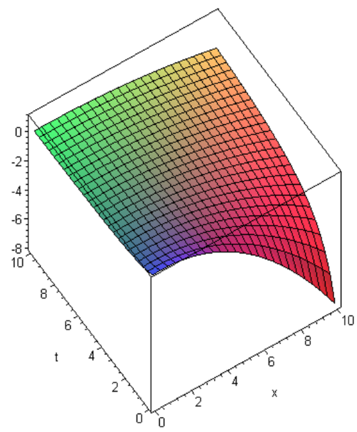

Fig. 6. Integral surface $y(x,t)$ in case of $C = 10$, $B = 10$.

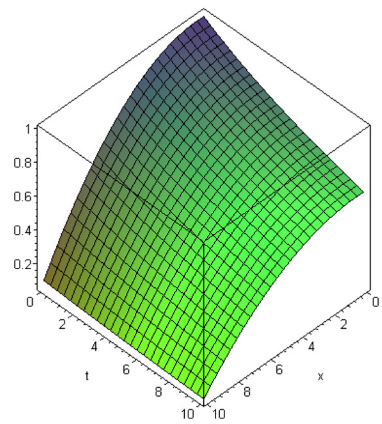

Fig. 7. Integral surface $y(x,t)$ in case of $C = 100$, $B = 10$**.**

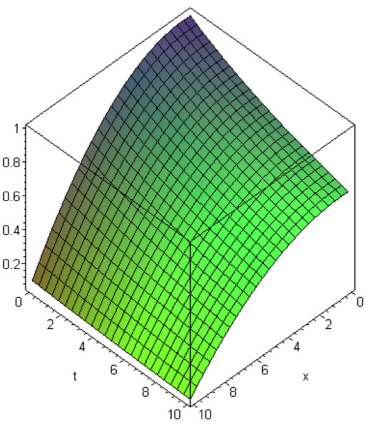

Fig. 8. Integral surface $y(x,t)$ in case of $C = 100$, $B = 100$.

It follows from the calculation:

1. A change in the determinative dimensionless parameters C and B can result in a radical rearrangement of the flow (see, for example, Fig. 2 and 8).

2. The assumption that elements AB and $B0x$ are impenetrable for the liquid (which is conventional in the theory of filtration) is introduced in order to use predeterminedly one-dimensional models. In this case the level height $h \geq 0$. The appearance of negative level height values in the chosen coordinate system does not contradict the physical sense of the formulated problem. However, this indicates the necessity of solving multidimensional tasks. It is interesting to note that the possibility of the appearance of negative   values follows even from the one-dimensional non-stationary Leybenson equation (see Fig. 2, 3 and 6).

3. A change of parameter   considerably affects the size and liquid level *h(x,t)* with respect to the chosen coordinate system.

4. Choosing parameters C and B allows to consider the initial distribution of height *h(0,t)* determining the further evolution of the system.

5. A change of parameter B (at fixed C) in the above calculations scarcely affects the system evolution. (Compare the calculations results shown in Fig. 7 and 8.)

In conclusion, note that flows in porous materials are not only a research subject of the classical theory of filtration. Calculation of thermal protection coatings of head parts of spacecrafts (in particular, reusable space systems of the Buran of Shuttle type) results in the necessity of finding self-consistent solutions of transfer equations for the cases of external non-viscous flow, the boundary layer of a reacting mixture of gases and a flow in the pores of a thermal protection coating, for example, graphite [6, 7].

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