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HYPERBOLIC MODEL OF NON -STATIONARY THERMAL CONDUCTIVITY*

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The article presents fundamentally new results on the analytical theory of thermal conductivity for hyperbolic transport models. The questions of the correct formulation of boundary value problems are considered. A technique for finding exact analytical solutions of a rather complex class of boundary value problems based on the method of Green's functions and operational calculus is developed.

Keywords: *transport model, a hyperbolic equation boundary, value problems, analytical solutions.*

Introduction

For historical reasons the linear gradient Fourier equation $\vec{q}(M,t) = -\lambda gradT(M,t)$ is the most commonly practically encountered model of heat conductivity in undeformed bodies. Along with the energy equation for isotropic solid bodies $c\rho\frac{\partial T(M,t)}{\partial t} = -div\vec{q}(M,t) + F(M,t)$ Fourier's law leads to an equation of parabolic type for non-stationary heat transfer of the form

$$\frac{\partial T(M,t)}{\partial t} = a\Delta T(M,t) + \frac{1}{c\rho}F(M,t),\ M \in D, t > 0 \tag{1}$$

and boundary value problems corresponding to (1) with the following initial and end conditions:

$$T(M,t)\big|_{t=0} = \Phi_0(M),\ M \in \overline{D}, \tag{2}$$

$$\beta_1\frac{\partial T(M,t)}{\partial n} + \beta_2 T(M,t) = \beta_3\varphi(M,t),\ M \in S, t > 0. \tag{3}$$

Here D is the finite or partially limited convex area of variation of $M(x,y,z)$, S is a piecewise smooth surface limiting area D, $\vec{n}$ is an external normal to S (a vector continuous in the points of S), $\Omega = (M \in D, t > 0)$ is a cylindrical area in phase space (x,y,z,t) with the basis D at $t = 0$. The parameters in (1)–(3) are thermophysical characteristics of the medium, constants in the range of temperatures not exceeding the transition points [1, 2]. The boundary functions in (1)–(3) belong to the class of functions $F(M,t) \in C^0(\overline{\Omega})$, $\Phi_0(M) \in C^1(\overline{\Omega})$,

$\varphi(M,t) \in C^0(S \times t \geq 0)$, the required solution is $T(M,t) \in C^2(\Omega) \cap C^0(\overline{\Omega})$; $grad_M T(M,t) \in C^0(\overline{\Omega})$, $\beta_1^2 + \beta_2^2 > 0$.

The author of [1] developed a series of analytical methods for finding exact solutions of boundary value problems (1)–(3), particularly, in the form of the following integrated representation at temporary and spatial nonuniformities in the initial problem formulation:

$$T(M.t) = \iiint_D \Phi_0(P) G(M,t,P,\tau)\big|_{\tau=0} dV_P + a\int_0^t \iint_S \left[G(M,t,P,\tau)\frac{\partial T(P,\tau)}{\partial n_P} - T(P,\tau)\frac{\partial G(M,t,P,\tau)}{\partial n_P}\right] d\tau d\sigma_P +$$
$$+\int_0^t \iiint_D \frac{1}{c\rho} F(P,\tau) G(M,t,P,\tau) d\tau dV_P \tag{4}$$

Here $G(M,t,P,\tau)$ is Green function for this area as the solution of a simpler problem for the uniform equation (1) with uniform boundary conditions of the same type as (3):

$$\frac{\partial G}{\partial t} = a\Delta_M G(M,t,P,\tau),\ \ M \in D, t > \tau, \tag{5}$$

$$G(M,t,P,\tau)\big|_{t=\tau} = \delta(M,P),\ (M,P) \in D, \tag{6}$$

$$\beta_1 \frac{\partial G(M,t,P,\tau)}{\partial n} + \beta_2 G(M,t,P,\tau) = 0,\ M \in S, t > \tau. \tag{7}$$

For limited areas D of the canonic type Green function G is given by

$$G(M,t,P,\tau) = G(M,t-\tau,P) = \sum_{n=1}^{\infty} \frac{\Psi_n(M)\Psi_n(P)}{\|\Psi_n\|^2} \exp\left[-\left(\sqrt{a}\gamma_n\right)^2 (t-\tau)\right], \tag{8}$$

where $\Psi_n(M)$ and γ_n^2 are eigenfunctions and eigenvalues of uniform problem

$$\begin{cases} \Delta\Psi(M) + \gamma^2\Psi(M) = 0, & M \in D, \\ \beta_1 \dfrac{\partial \Psi(M)}{\partial n} + \beta_2 \Psi(M) = 0, & M \in S, \end{cases} \tag{9}$$

corresponding to (1)–(3),

$\|\Psi_n\|^2 = \iiint_D \Psi_n^2(M) dV_M$ is the square of the norm or the eigenfunctions.

Here δ (z) is Dirac delta function. On the basis of the solution of spectral problems (9) Kartashov tables were developed in [1–3]. (This is a conventional term in the scientific and educational literature.) The tables use a rather simple scheme and allow writing the exact analytical solution of the thermal problem (1)–(3) in the Cartesian, cylindrical and spherical systems of coordinates in the form of Fourier – Hankel series with improved convergence up to the area boundary. This is very convenient for engineering calculations when determining the thermophysical characteristics of materials, determining the warming up time of samples, establishing the time of transition to the stationary phase when heating or cooling, etc.

Despite some paradoxes encountered when using model representations (1)–(4) [the lack of inertia of heat conductivity in Fourier's law and, as a result, the conclusion following from (4) about the infinite rate of heat transfer; the singular character of the heat flow and of the speed of the isotherms motion in the area $x > 0$, $t > 0$], the latter fact does not limit the scope of application of boundary value problems (1)–(3) as a subject of the practically immeasurable number of studies covering new substantial mathematical objects and the increasing number of the most diverse analytical methods giving exact analytical solutions of thermal problems (1)–(3) [1–3].

Since 1930s studies on heat transfer in liquids, gases and solid bodies with a finite rate have been developed ([4] and references in [4]). In 1940 L. Tisza and in 1941 irrespective of him L. Landau demonstrated the possibility of the existence of a finite rate υ_T of heat transfer in liquid helium II. (This was named second sound – SS.) These studies were continued by V. Peshkov (1946), who showed that SS can exist in pure solid bodies. (It was found that $\upsilon_T \approx 720$ m/s in crystals at $T = 3.4^0 K$). J. Ward and J. Wilks (1952) suggested a formula for evaluating υ_T in solid bodies with the use of measurable macroscopic parameters ($\upsilon_T \approx \frac{\upsilon_P}{\sqrt{3}}$ in crystals, in metals, υ_P is the sound speed). R. Dingle (1952) studied heat transfer in dielectrics, superconductors and ferromagnetics, F. London (1954) – in metals and glasses, C. Ackerman, B. Bertman, H. Fairbank, R. Guyer (1966) – in crystalline helium. M. Chester (1963) considered the second sound in solid bodies from the macroscopic point of view and indicated that the equation of heat transfer must include a summand containing velocity υ_T on the basis of results obtained by Maxwell – the first one who introduced inertia into the transfer equations – and Cattaneo, who suggested a version of Fourier law with a relaxation member of heat flow. In 1965 S. Kaliski [5] established the generalized law of heat conductivity by introducing the rate of heat flow change – thermal inertia – into Onsager's principle. Practically at the same time (1965) and independently for isotropic bodies A.V. Lykov [3] established the generalized law of heat and mass transfer as a hypothesis of finite rates of heat and mass transfer for heat and moisture transfer in capillary-porous bodies.

Let us write the generalized system of Onsager equations in the form

$$\vec{J}(M,t) = L_r \frac{\partial \vec{J}(M,t)}{\partial t} + \sum_{k=1}^{N} \left[L_k \vec{X}_k(M,t) + L_k^{'} \frac{\partial \vec{X}_k(M,t)}{\partial t} \right],$$

where $\vec{J}(M,t)-$is the flow of substance (heat, mass etc) in area D at $t>0; \vec{X}_k(M,t)-$are thermodynamic driving forces (gradients of temperature, concentration etc); $L_r, L_k, L_k^{'}-$are kinetic coefficients (constant phenomenological coefficients of transfer). If we neglect the time derivative of driving force $\vec{X}_k$ assuming that $\vec{J}(M,t)=\vec{q}(M,t)-$is the vector of heat flow density, and $L_r=-\tau_r, \vec{X}_k = qradT(M,t), L_k=-\lambda$ (the medium heat conductivity) ($N=1$), we obtain the following generalized law of heat conductivity for solid bodies:

$$\vec{q}(M,t) = -\lambda gradT(M,t) - \tau_r \frac{\partial \vec{q}(M,t)}{\partial t}, \tag{10}$$

where τ_r is the relaxation time of the heat flow connected with the heat transfer rate by the relation $\upsilon_T = \sqrt{a/\tau_r}$. For metals $\tau_r \approx (10^{-14}-10^{-11})$ s. For amorphous bodies like inorganic glass and polymers having a complex structure $\tau_r \approx (10^{-11}-10^{-5})$ s. (For inorganic glass $\tau_r \approx 10^{-7}$ s, for organic glass $\tau_r \approx 10^{-11}$ s.) For nitrogen $\tau_r \approx 10^{-9}$ s. Experimental measurement of τ_r in many cases is not possible. The rate of heat transfer for steel $\upsilon_T = 1800$ m/s (the speed of sound $\upsilon_P = 6100$ m/s). For aluminum $\upsilon_T = 2930$ m/s ($\upsilon_P = 6260$ m/s). For inorganic glass $\upsilon_T = 2*10^6$ m/s ($\upsilon_P = 4.5*10^3$ m/s). For nitrogen $\upsilon_T = 150$ m/s, and for gases in the conditions of rarified supersonic stream the influence of the final rate of heat transfer on heat exchange becomes noticeable. Similar influence can show itself also at very low temperatures (for example, in liquid helium $\upsilon_T = 19$ m/s at $T = 1.4^0 K$) and even at ordinary temperatures in solid bodies, when a small period of time in a non-stationary process is considered [6]. The energy equation for isotropic solid bodies and relation (10) lead to the equation of heat conductivity of the hyperbolic type:

$$\frac{\partial T(M,t)}{\partial t} = a\Delta T(M,t) - \tau_r \frac{\partial^2 T(M,t)}{\partial t^2} + \frac{\tau_r}{c\rho}\left[\frac{\partial F(M,t)}{\partial t} + \frac{1}{\tau_r}F(M,t)\right], \quad M \in D, t>0. \tag{11}$$

and to corresponding boundary value problems of thermal conductivity for equation (11) of the generalized form.

Equation (11) was also obtained by A.S. Predvoditelev. However, he proceeded from other ideas, namely, from the analysis of the displacement velocities of isothermal surfaces with the use of Riemann representation, i. e., he completely rejected relaxation formula (10).

Systematic publications on hyperbolic models of transfer can be associated with late 1960s. The authors of [6] carried out one of the first works using model representations for equation (11) in order to describe the thermal reaction T(z, t) of an elastic half-space $z>0$, $t>0$ at thermal heating of its boundary (Derichlet's boundary conditions). Having analyzed the analytical solution of a similar problem at boundary temperature $T(0,t) = T_0 + \left[\frac{T_C - T_0}{t_0}\right]\left[t - \eta(t-t_0)(t-t_0)\right]$, (η(z) – is Heaviside function), A.V. Lykov justified the physical sense of the final rate of heat transfer: it is the time derivative of heat penetration depth. The generalized problems of transfer considerably differ from the classical ones (1)–(3): they are more difficult when finding analytical solutions of these problems. This is the reason for the very insignificant progress in finding exact analytical solutions of boundary value problems for equation (11) and mostly for semi-limited area $z > 0$, $t > 0$ (in the main statement) at constant boundary functions and zero boundary conditions [7, 8]. For areas of the canonic type (an infinite plate, a continuous or hollow cylinder, a continuous or hollow sphere, etc.) exact solutions of hyperbolic models of transfer are still unknown, and this problem essentially remains open including the correct statement of boundary value problems for the hyperbolic equations. This publication is devoted to all these questions.

1. Boundary conditions for the hyperbolic equation of heat conductivity. The initial conditions for equation (11) can be written as functions of the general form

$$T(M,t)\big|_{t=0} = \Phi_0(M),\quad M \in \overline{D}, \tag{12}$$

$$\frac{\partial T(M,t)}{\partial t}\Big|_{t=0} = \Phi_1(M),\quad M \in \overline{D}. \tag{13}$$

Depending on the form of boundary conditions the following problems can be considered for equation (11):

the Dirichlet boundary value problem (first-type boundary condition)

$$T(M,t) = \varphi_C(M,t),\quad M \in S, t > 0; \tag{14}$$

the Neumann boundary value problem (generalized second-type boundary condition)

$$-\lambda \frac{\partial T(M,t)}{\partial t} = \left(1 + \tau_r \frac{\partial}{\partial t}\right)\varphi_C(M,t),\quad M \in S, t > 0; \tag{15}$$

the mixed boundary value problem (generalized third-type boundary condition)

$$\frac{\partial T(M,t)}{\partial n} = -h\left(1 + \tau_r \frac{\partial}{\partial t}\right)\left[T(M,t) - \varphi_C(M,t)\right],\quad M \in S, t > 0. \tag{16}$$

The functions in (12)–(16) are of the function class $\Theta(M,t) \in C^2(\Omega)$; $\Phi_0(M) \in C^1(\overline{D})$; $\Phi_1(M) \in C^0(\overline{D})$; $\varphi_C(M,t) \in C^1(\Omega)$; the required solution $T(M,t) \in C^2(\Omega) \cap C^1(\Omega)$.

Generalized boundary conditions (15)–(16) are written in the differential form allowing another form, the integrated one. The latter is possible only under certain conditions imposed on the boundary functions in (12)–(16). Thus, for (15) it is possible to write equivalent second-type boundary condition

$$-\frac{1}{\tau_r}\int_0^t \frac{\partial T(M,t)}{\partial n}\exp\left[-\frac{t-\tau}{\tau_r}\right]d\tau = \frac{1}{\lambda}\varphi_C(M,t),\ M\in S, t\geq 0, \tag{17}$$

if equality $\varphi_C(M,0)=0$, $M\in S$ is true. For (16) we have equivalent integral form

$$\frac{1}{\tau_r}\int_0^t \frac{\partial T(M,t)}{\partial n}\exp\left[-\frac{t-\tau}{\tau_r}\right]d\tau = -h\left[T(M,t)-\varphi_C(M,t)\right],\ M\in S, t\geq 0, \tag{18}$$

if equality $\varphi_C(M,0)=\Phi_0(M)$, $M\in \overline{D}$ is true.

2. Analytical solutions. The second- and third-type boundary conditions in the form of (main) (15)–(16) show that the corresponding spectral problems cannot be solved for the second and third boundary value problems. So, Kartashov's tables of integrated Fourier-Hankel transformations developed on the basis of solving these problems cannot be applied (in the Cartesian, cylindrical and spherical systems of coordinates) when finding analytical solutions. Therefore, exact analytical solutions of the second-type, third-type and mixed boundary value problems for areas of canonic type are still not found.

Let us consider one of such problems of applied thermomechanics, which is of interest for the theory of heat shock [2]. Let us imagine that there is a plane-parallel elastic uniform isotropic layer of finite thickness l with boundaries free from strain. The layer occupies area $0\leq x\leq l, -\infty < y, z < +\infty.$ in the Cartesian coordinates. Heat exchange occurs through a surface of the layer $x=l$ with the external medium, the temperature of which changes at the initial timepoint from T_0 to T_c ($T_c > T_0$) remaining constant afterwards, and surface $x=0$ is maintained at temperature T_0. When $t\leq 0$, the temperature of the layer is equal to T_o, and the rate of heating is assumed to be equal to zero. Let us write the mathematical model of the problem for equation (11) with respect to temperature function $T(x,t)$ (in the absence of internal sources) in dimensionless variables assuming that

$$z = x/l; F_0 = at/l^2; Bi = hl; c* = a\tau_r/l^2; W(z,F_0) = \frac{T(x,t)-T_0}{T_c-T_0}.$$

We have the following hyperbolic model of non-stationary heat conductivity:

$$\frac{\partial W}{\partial F_0} = \frac{\partial^2 W}{\partial z^2} - c*\frac{\partial^2 W}{\partial F_0^2}, \qquad 0<z<1, F_0>0, \tag{19}$$

$$W\Big|_{F_0=o} = \frac{\partial W}{\partial F_0}\Big|_{F_0=0} = 0, \qquad 0\leq z\leq 1, \tag{20}$$

$$W\big|_{z=0}=0, \qquad F_0>0, \tag{21}$$

$$\frac{\partial W}{\partial z}\Big|_{z=1}=-Bi\left(1+c*\frac{\partial}{\partial F_0}\right)\left(W\big|_{z=1}-1\right), \qquad F_0>0. \tag{22}$$

In Laplace's image domain $\bar{W}(z,p)=\int_0^{\infty}W(z,F_0)\exp(-pF_0)dF_o$ the solution of problem (19)–(22) is given by

$$\bar{W}\ (z,p)=\frac{Bish\sqrt{\gamma}z}{\sqrt{\gamma}(pch\sqrt{\gamma}+Bi\sqrt{\gamma}sh\sqrt{\gamma})}, \tag{23}$$

where $\sqrt{\gamma}=\sqrt{c*p^2+p}$. Expressions of type (23) are typical images for hyperbolic models of transfer after applying Laplace transformation to (19)–(22). Transition to the original in (23) requires long transformations. So, we will discuss only the main points of the transition. Let us use equation [1]

$$\gamma_1 sh\sqrt{\gamma}+\gamma_2 ch\sqrt{\gamma}=\frac{\gamma_1+\gamma_2}{2}\exp(\sqrt{\gamma})\left[1-\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\exp(-2\sqrt{\gamma})\right],$$

to transform equation (23) to the form

$$\bar{W}(z,p)=Bi\ \sum_{n=o}^{\infty}\frac{(Bi\sqrt{\gamma}-p)^n}{(Bi\sqrt{\gamma}+p)^{n+1}}\sum_{k=1}^{2}(-1)^{k-1}\frac{1}{\sqrt{\gamma}}\exp\left[-\alpha_{k,n}(z)\sqrt{\gamma}\right]=\sum_{n=0}^{\infty}\frac{1}{p}\frac{(\sqrt{c*+\frac{1}{p}}-1/Bi)^n}{\sqrt{c*+\frac{1}{p}}+1/Bi)^{n+1}}\times$$

$$\times\sum_{k=1}^{2}(-1)^{k-1}\frac{1}{\sqrt{c*}\sqrt{p(p+1/c*)}}\exp[-\alpha_{kn}(z)\sqrt{c*}\sqrt{p(p+1/c*)}], \tag{24}$$

where $\alpha_{1n}(z)=(2n+1)-z;\alpha_{2n}=(2n+1)+z.$

The original of expression

$$\bar{W}_1(p)=\frac{1}{p}\frac{(\sqrt{c*+\frac{1}{p}}-1/Bi)^n}{(\sqrt{c^*+\frac{1}{p}}+1/Bi)^{n+1}}$$

is found by using the following relations of operational calculus:

$$\bar{W}_1(p)=\frac{1}{p}\bar{F}(\frac{1}{p})\leftarrow\int_0^{\infty}J_0(2\sqrt{F_0\tau})F(\tau)d\tau,$$

$$\bar{F}(p)=\frac{(\sqrt{c*+p}-1/Bi)^n}{(\sqrt{c*+p}+1/Bi)^{n+1}}\leftarrow\exp(-c*F_0)F_1(F_0)=F(F_0);$$

$$\bar{F}_1(p)=\frac{(\sqrt{p}-1/Bi)^n}{(\sqrt{p}+1/Bi)^{n+1}}=\bar{F}_2(\sqrt{p})\leftarrow\int_0^{\infty}\frac{y}{2\sqrt{\pi}F_0^{3/2}}\exp(-\frac{y^2}{4F_0})F_2(y)dy;$$

$$\bar{F}_2(p)=\frac{(p-1/Bi)^n}{(p+1/Bi)^{n+1}}\leftarrow \exp(-\frac{F_0}{Bi})L_n(\frac{2F_0}{Bi})=F_2(F_0)$$

$$F_1(F_0)=\int_0^\infty \frac{y}{2\sqrt{\pi}F_0^{3/2}}\exp(-\frac{y^2}{4F_0}-\frac{y}{Bi})L_n(\frac{2y}{Bi})dy$$

$$F(F_0)=\frac{1}{2\sqrt{\pi}F_0^{3/2}}\exp(-c*F_0)\int_0^\infty y\exp(-\frac{y^2}{4F_0}-\frac{y}{Bi})L_n(\frac{2y}{Bi})dy.$$

Thus,

$$W_1(F_0)=\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{\tau^{3/2}}J_0(2\sqrt{F_0\tau})\exp(-c*\tau)d\tau\int_0^\infty y\exp(-\frac{y^2}{4\tau}-\frac{y}{Bi})L_n(\frac{2y}{Bi})dy. \qquad (25)$$

Here $J_0(z)-$ is the first-type Bessel's function of zero order; $L_n(z)-$ are Laguerre's polynomials.

The original of expression

$$\bar{W}_{k,n}(z,p)=\frac{1}{\sqrt{c*}\sqrt{p(p+1/c*)}}\exp\left[-\alpha_{k,n}(z)\sqrt{c*}\sqrt{p(p+1/c*)}\right]$$

is written as follows:

$$W_{k.n}(z,F_0)=\frac{1}{\sqrt{c*}}\exp(-\frac{F_0}{2c*})I_0(\frac{1}{2c*}\sqrt{F_0^2-c*\alpha_{k,n}^2(z)})\eta\left[F_0-\sqrt{c*}\alpha_{k,n}(z)\right] \qquad (26)$$

Here $I_0(z)$ is the modified Bessel's function. Now, having (25) and (26), let us use the convolution theorem to find the exact solution of problem (19)–(22) in the following form:

$$W(z,F_0)=\sum_{n=0}^{\infty}\sum_{k=1}^{2}(-1)^{k-1}\int_0^{F_0}W_1(\tau')W_{k,n}(z,F_0-\tau')d\tau'. \qquad (27)$$

However, solution (27) can have another functional design, and the features of hyperbolic models of transfer are distinctly shown here.

Let us consider briefly this matter.

The denominator of function (23) written as

$$\frac{\bar{W}(z,p)}{Bi}=\frac{\bar{f}_1(z,p)}{\bar{f}_2(p)}=\frac{shz\sqrt{\gamma}/\sqrt{\gamma}}{pch\sqrt{\gamma}+Bi\sqrt{\gamma}sh\sqrt{\gamma}} \qquad (28)$$

in plane p has an infinite amount of zeroes (poles) determined by equation $p_n ch\sqrt{\gamma_n}+Bi\sqrt{\gamma_n}sh\sqrt{\gamma_n}=0$,

whence it follows that

$$\sqrt{\gamma_n}=-i\mu_n^2, c*p_n^2+p_n+\mu_n^2=0, p_n=Bi\mu_n tg\mu_n,$$

and

$$p_n=-(1/2c*)\pm i\omega_n, \omega_n=\frac{1}{\sqrt{c*}}\sqrt{\mu_n^2-1/(4c*)},$$

the value $p=0$ as well is a pole of function (28), and numbers $\mu_n > 0$ are roots of equation

$$Bi^2 c* \mu_n tg^2 \mu_n + Bitg\mu_n + \mu_n = 0. \tag{29}$$

Using Vaschenko-Zakharchenko's expansion theorem [1] in the form

$$\frac{W(z,F_0)}{Bi} = \frac{\bar{f}_1(z,0)}{\bar{f}_2^{'}(0)} + \sum_{n=1}^{\infty} \frac{\bar{f}_1(z,p_n)}{\bar{f}_2^{'}(p_n)} \exp(p_n F_0)$$

after long transformations we find the original of transform (28) – the exact analytical solution of problem (19)–(22) in a form differing from (26):

$$W(z,F_0) = \frac{Biz}{1+Bi} +$$

$$+2Bi \sum_{n=1}^{\infty} \frac{(\varphi_{1n}\cos\omega_n F_0 + \varphi_{2n}\sin\omega_n F_0)\cos\mu_n \sin\mu_n z}{\varphi_{1n}^2 + \varphi_{2n}^2} \exp(-\frac{F_0}{2c*}), \tag{30}$$

where

$$\varphi_{1n} = \mu_n \cos^2 \mu_n, \varphi_{2n} = (\mu_n + \sin\mu_n \cos\mu_n) Bic* \omega_n.$$

Applying the above approaches it is possible to obtain exact analytical solutions of thermal problems for equation (11) in area $x \in [0,l], t \geq 0$ with boundary conditions of any type. However, if equation (11) and boundary conditions (12)–(18) contain sufficiently general temporal and spatial nonuniformities, technical difficulties of solving can become insuperable. These difficulties can be avoided by uniting the method of Green function for hyperbolic models of transfer with the operational method (which implies initial finding of the corresponding Green function – a simpler problem, and then finding the required solution by means of its integrated representation – an analog of (4)–(8) for equation (11)). This method was developed by the author of [9]. Thus, in case of the first boundary value problem in (14) or the second boundary value problem in (15) Green function is given by

$$G(M,t-\tau,P) = \sum_{n=1}^{\infty} \frac{\Psi_n(M)\Psi_n(P)}{\|\Psi_n\|^2 \sqrt{(\sqrt{a\tau_r}\gamma_n)^2 - \frac{1}{4}}} \sin\left[\sqrt{(\sqrt{a\tau_r}\gamma_n)^2 - \frac{1}{4}}(\frac{t-\tau}{\tau_r})\right] \exp(-\frac{t-\tau}{2\tau_r}), \tag{31}$$

where $\Psi_n(M)$ and γ_n^2 are eigenfunctions and eigenvalues of spectral task (9) according to the first- or second-type boundary conditions. Having result (31) and the integral relation in [9] it is easy to write the analytical solution of equation (11) with sufficiently general boundary conditions. Note that results (26), (30) and (31) are apparently presented in press for the first time.

The above ratios clearly demonstrate the difficulties of finding analytical solutions of hyperbolic models of transfer. In this respect a lot of work is expected to be done for the development of the corresponding direction of the analytical theory of heat conductivity of solid bodies.

Conclusions

Fundamentally new results of the analytical theory of heat conductivity of solid bodies relating to hyperbolic models of transfer are presented. It is shown that the method of Green function in combination with the operational method enables obtaining exact the analytical solutions of the problems in the integrated form containing all the nonuniformities in the initial formulation of the problems.

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