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MATHEMATICS METHODS AND INFORMATION SYSTEMS IN CHEMICAL TECHNOLOGY
МАТЕМАТИЧЕСКИЕ МЕТОДЫ И ИНФОРМАЦИОННЫЕ СИСТЕМЫ В ХИМИЧЕСКОЙ ТЕХНОЛОГИИ

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Originals of operating images for generalized problems of unsteady heat conductivity

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A series of operating (Laplace) non-standard images, the originals of which are absent in well-known reference books on operational calculus, are considered. By reducing one of the basic images to the Riemann-Mellin contour integral for the modified Bessel functions and analyzing the corresponding inversion formula using the approaches of the complex variable function theory, an analytical form of the original original is found, which is abrupt in nature with a break point. It is shown that analytical solutions of the corresponding mathematical models using the found originals have a wave character, which is expressed by the presence of the Heaviside step function in the solutions. The latter means that at any time there is a region of physical disturbance to the point of discontinuity and an unperturbed area after the point of discontinuity. The images studied are included in the operational solutions of mathematical models in many areas of applied mathematics. physics, thermomechanics, thermal physics, in particular in the theory of thermal shock of viscoelastic bodies, in the study of the thermal reaction of solids based on the classical Maxwell-Cattaneo-Lykov-Vernott phenomenology, taking into account the final rate of heat propagation. These models are needed to study the thermal reaction of relatively new consolidated structurally sensitive polymeric materials in structures exposed to high-intensity external influences. The analytical relations obtained for the originals and the original improper integrals resulting from them, containing combinations of Bessel functions, can be used in the general methodology of constructing and applying various mathematical models in a wide range of external influences on materials in many fields of science and technology.

Keywords: *originals of operational images, hyperbolic models of unsteady heat conduction, thermal shock.*

Оригиналы операционных изображений для обобщенных задач нестационарной теплопроводности

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Рассмотрена серия операционных (по Лапласу) нестандартных изображений, оригиналы которых отсутствуют в известных справочниках по операционному исчислению. Путем сведения одного из базовых изображений к контурному интегралу Римана-Меллина для модифицированных функций Бесселя и анализа соответствующей формулы обращения с *использованием подходов теории функций комплексного переменного установлен аналитический вид искомого оригинала, имеющего скачкообразный характер с точкой разрыва. Показано, что аналитические решения соответствующих математических моделей с использованием найденных оригиналов имеют волновой характер, что выражается наличием в решениях ступенчатой функции Хевисайда. Последнее означает, что в любой момент времени существует область физического возмущения до точки разрыва и невозмущенная область после точки разрыва. Изученные изображения входят в операционные решения математических моделей во многих областях прикладной математики, физики, термомеханики, теплофизики, в частности в теории теплового удара вязкоупругих тел, при изучении тепловой реакции твердых тел на основе классической феноменологии Максвелла-Каттанео-Лыкова-Вернотта с учетом конечной скорости распространения теплоты. Указанные модели необходимы для изучения термической реакции сравнительно новых консолидированных структурно-чувствительных полимерных материалов в конструкциях, подверженных высокоинтенсивным внешним воздействиям. Полученные для оригиналов аналитические соотношения и вытекающие из них оригинальные несобственные интегралы, содержащие комбинации функций Бесселя, могут быть использованы в общей методологии построения и применения разнообразных математических моделей в широком диапазоне внешних воздействий на материалы во многих областях науки и техники.*

Ключевые слова: *оригиналы операционных изображений, гиперболические модели нестационарной теплопроводности, тепловой удар.*

Introduction

Modern structural materials, which are a combination of micro- or nanostructured elements, are often called structurally sensitive materials. The creation of such materials based on nanotechnology is an important direction in the development of modern materials science. Such materials have unique physico-mechanical properties that allow them to be used effectively in structures subject to high-intensity external influences [1, 2]. An important step in the creation and use of these kinds of materials is the construction of appropriate mathematical models to describe their behavior in a wide range of changes in external loads. The general methodology for constructing and studying such models is still far from complete and requires further development. This applies primarily to the mathematical models of a number of physical processes while taking into account spatio-temporal nonlocality.

Classical phenomenological models of transport processes and other phenomena, such as Fourier heat, Nernst mass, Ohm electricity, Newton and Hook voltages, are based on the principle of local thermodynamic equilibrium and the continuous medium hypothesis. The differential equations derived from them for the corresponding physical quantities are local, that is, they do not take into account local non-equilibrium processes; in the process of derivation, an infinite propagation velocity of disturbances is incorporated into them. Moreover, the functions describing these processes are smooth functions of coordinates and time. However, the propagation velocity of the potentials of any physical fields cannot take infinite values. In a real body, the process of their change occurs with a certain delay in time according to the relaxation properties of the material, which are taken into account by relaxation coefficients. Such processes exist in reality. They have so-called front surfaces, passing through which makes the temperature function and its derivatives acquire a discontinuity [3, 4]. These functions are described by hyperbolic differential operators. They include high-intensity non-stationary processes, the flow time of which is comparable to the relaxation time. Examples include heating materials with short laser pulses (duration varies from nano- to femtoseconds); heating processes with friction at a high speed; during a thermal shock; local heating during dynamic propagation of a crack in a transonic mode, etc.

Taking into account the local non-equilibrium embedded in the Maxwell-Cattaneo-Lykov-Vernotte relation for the heat flux (in the one-dimensional case)

$$q(x,t) = -\lambda \frac{\partial T(x,t)}{\partial x} - \tau_r \frac{\partial q(x,t)}{\partial t} \qquad (1)$$

together with the energy equation $c\rho \partial T / \partial t = -\partial q / \partial x$ lead to the heat hyperbolic type equation [5]

$$\frac{\partial T(x,t)}{\partial t} = a \frac{\partial^2 T(x,t)}{\partial x^2} - \tau_r \frac{\partial^2 T(x,t)}{\partial t^2} \qquad (2)$$

and the corresponding boundary value problems of a generalized type [6]. In this case, τ_r signifies the thermal relaxation time (a measure of the inertia of the heat flux) associated with the rate of heat propagation v_T by the relation $\tau_r = a / v_T^2$ (a – thermal diffusivity). When $v_T \to \infty$, the magnitude $\tau_r \to 0$ and relations (1)–(2) respectively lead to the classical phenomenological law of Fourier heat transfer and the parabolic-type heat equation, which underlies an almost unlimited number of studies on non-stationary heat transfer. The generalized transport problems for equation (2) differ significantly from the classical ones, with it being more difficult to find their analytical solutions. The specificity of such problems lies in the relative simplicity of the original mathematical models and the difficulties of solving them in an analytically closed form. This results in very little success in finding their exact analytical solutions. The main method for solving boundary-value problems of a generalized type for partially bounded domains is the operational one, which leads to complex functional constructions of the Karslow-Jäger type [7] in analytic solutions in the Laplace image space $\overline{T}(x,p) = \int\limits_0^\infty \exp(-pt)T(x,t)dt$. The originals of the aforementioned Karslow-Jäger type constructions do not appear in well-known reference books on operational calculus. Serious computational difficulties arise along this path. The aim of this publication is to consider a series of non-standard images and their originals. In addition to the generalized problems of non-stationary transfer (heat and mass), such images also arise in the description of electric transmission lines, in the study of transient modes of electrical circuits (the propagation of electrical disturbances along the transmission line); in the thermal shock theory of viscoelastic bodies, etc. Let us dwell on the generalized problem for equation (2) in the region $x > 0$, $t > 0$ under the initial condition

$$T(x,t)\big|_{t=0} = T_0, x \ge 0 \tag{3}$$

and boundary conditions of either the first kind (temperature heating or cooling)

$$T(x,t)\big|_{x=0} = T_c, t > 0, \tag{4}$$

or the second kind (thermal heating or cooling)

$$\frac{1}{\tau_r}\int\limits_0^t \frac{\partial T(x,\tau)}{\partial x}\Big|_{x=0} \exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau = -\frac{1}{\lambda}q_0, t > 0, \tag{5}$$

or of the third kind (heating or cooling by the environment)

$$\frac{1}{\tau_r}\int\limits_0^t \frac{\partial T(x,t)}{\partial x}\Big|_{x=0} \exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau = \\ = h\left[T(x,t)\big|_{x=0} - T_c\right], t > 0, \tag{6}$$

as well as the constraint condition (in all three cases)

$$\left|T(x,t)\right| < \infty, x \ge 0, t \ge 0. \tag{7}$$

It should be noted that questions of the correct formulation of the boundary conditions for the hyperbolic type equation (2) were considered by the author in [7].

Let consider the next theory: the originals for non-standard images.

Inversion theorems for images

Consider a series of images of the form

$$\overline{f}(p)\exp\left[-x\sqrt{(p+2\alpha)(p+2\beta)}\right],$$

or

$$\begin{cases} \overline{f}(p)\exp\left[-x\overline{\mu}(p)\right] \\ \overline{\mu}(p) = \sqrt{(p+2\alpha)(p+2\beta)}, \end{cases} \tag{8}$$

where $\overline{f}(p)$ – are various combinations of rational and irrational functions of the argument p.

Initially we examine Riemann-Mellin type integral

$$Y_1(x,t) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{\overline{\mu}(p)}\exp\left[pt - x\overline{\mu}(p)\right]dp. \tag{9}$$

The use of the representation of the Bessel function of an imaginary argument $I_n(z)$ in the form of the integral [8]

$$\left(\frac{2}{z}\right)^n I_n(z) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{u^{n+1}}\exp\left(u+\frac{z^2}{4u}\right)du \tag{10}$$

and give (9) to a form similar to equation (10). For this, suppose [7]:

$$(p+2\alpha)^{1/2} + (p+2\beta)^{1/2} = \xi^{1/2} \\ (p+2\alpha)^{1/2} - (p+2\beta)^{1/2} = 2\sigma\xi^{-1/2}, \tag{11}$$

the equation leads to

$$p=\frac{1}{4}(\xi+\frac{4\sigma^2}{\xi}-4\rho),\sqrt{(p+2\alpha)(p+2\beta)}=$$
$$=\frac{1}{4}(\xi-\frac{4\sigma^2}{\xi}), \quad (12)$$

$$\frac{d\xi}{\xi}=\frac{dp}{\sqrt{(p+2\alpha)(p+2\beta)}}. \quad (13)$$

Here, $\rho=\alpha+\beta, \sigma=\alpha-\beta$. Following this, the integral (9) is transformed by replacing the variable (13). In this case, the straight line ($\gamma-i\infty,\gamma+i\infty$) in the plane p is transformed into a line in the plane ξ. This line is not straight, but by Cauchy's theorem it can be deformed into a line, which is described as ($\gamma'-i\infty,\gamma'+i\infty$). Now, the integral (9) takes the form of

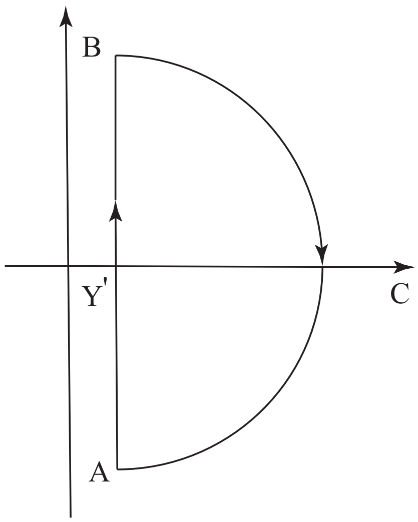
$$Y_1(x,t)=$$
$$=\frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{d\xi}{\xi}\exp\left[-\rho t+\frac{1}{4}\xi(t-x)+\frac{\sigma^2}{\xi}(t+x)\right] \quad (14)$$

If $t>x$, then, assuming in (14) $(\xi/4)(t-x)=u$ and $n=0$ from equation (10) we can derive:

$$Y_1(x,t)=\exp(-\rho t)I_0(\sigma\sqrt{t^2-x^2}),t>x. \quad (15)$$

If $t<x$, consider the integral (14), taken along a closed contour shown in Figure, which consists of a part of the contour $(\gamma'-i\infty,\gamma'+i\infty)$ and the arc of the circle with the radius R and a center at the start of the coordinates. The integrand function in (14) is regular inside the contour and on the boundary, and it does not contain any poles inside the contour. Afterwards, using the Cauchy theorem, the integral along this contour is equal to zero. It can be demonstrated that for $R\to\infty$ the integral along the arc of the circle turns into zero. Thus, this leads to the following result

$$Y_1(x,t)=0 \text{ for } t<x. \quad (16)$$



The contour for calculating the integral (14).

Finally, it is possible to derive

$$\frac{1}{\bar\mu(p)}\exp\left[-x\bar\mu(p)\right]\overset{*}{\underset{*}{\longleftarrow}}\exp(-\rho t)I_0(\sigma\sqrt{t^2-x^2})\eta(t-x), \quad (17)$$

where $\eta(t)$ is the Heaviside function. Further applying the convolution theorem, one can find:

$$\frac{1}{\bar\mu(p)}\exp\left[-x\bar\mu(p)\right]\bar f(p)\overset{*}{\underset{*}{\longleftarrow}}\int_0^t f(t-\tau)\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})\eta(\tau-x)d\tau=$$
$$=\begin{cases}\int_x^t f(t-\tau)\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})d\tau, & t>x,\\ 0, & t<x,\end{cases} = \quad (18)$$
$$=\left[\int_x^t f(t-\tau)\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})d\tau\right]\eta(t-x).$$

Differentiating (18) with respect to x leads to:

$$\exp\left[-x\overline{\mu}(p)\right]\overline{f}(p) \xleftarrow{*} \begin{cases} f(t-x)\exp(-\rho x)+\sigma x\int_x^t f(t-\tau)\exp(-\rho\tau)\dfrac{I_1(\sigma\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}d\tau, t>x. \\ 0, \qquad t<x. \end{cases} \tag{19}$$

Assuming in (19) that $\overline{f}(p)=1$, then $f(t)=\delta(t)$ is the Dirac function. Thus, from (19) we acquire:

$$\exp\left[-x\overline{\mu}(p)\right] \xleftarrow{*} \begin{cases} \exp(-\rho x)\delta(t-x)+\sigma x\exp(-\rho x)\dfrac{I_1(\sigma\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}}, t>x, \\ 0, \qquad t<x. \end{cases} \tag{20}$$

Suppose now that in (19) $\overline{f}(p)=\dfrac{1}{p}, f(t)=1$. We can derive the following:

$$\frac{1}{p}\exp\left[-x\overline{\mu}(p)\right] \xleftarrow{*} \begin{cases} \exp(-\rho x)+\sigma x\int_x^t \exp(-\rho\tau)\dfrac{I_1(\sigma\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}d\tau, t>x, \\ 0, \qquad t<x. \end{cases} \tag{21}$$

Assuming $\overline{f}(p)=\dfrac{1}{p}, f(t)=1$, in (18) then

$$\frac{1}{p\overline{\mu}(p)}\exp\left[-x\overline{\mu}(p)\right] \xleftarrow{*} \begin{cases} \int_x^t \exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})d\tau, t>x, \\ 0, \qquad t<x. \end{cases} \tag{22}$$

Based on that, we can find the original of the following image:

$$\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-x\overline{\mu}(p)\right]\overline{f}(p)=\frac{p+2\beta}{\overline{\mu}(p)}\exp\left[-x\overline{\mu}(p)\right]\overline{f}(p)=\left\{p\left[\frac{1}{\overline{\mu}(p)}\exp(-x\overline{\mu}(p))\right]\right\}\overline{f}(p)+$$

$$+2\beta\left\{\frac{1}{\overline{\mu}(p)}\exp\left[-x\overline{\mu}(p)\right]\right\}\overline{f}(p)\xleftarrow[*]{*}\int_0^t f(t-\tau)\frac{d}{d\tau}\left[\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})\right]\eta(\tau-x)d\tau+$$

$$+\int_0^t f(t-\tau)\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})\delta(\tau-x)d\tau+2\beta\int_x^t f(t-\tau)\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})d\tau=$$

$$=f(t-x)\exp(-\rho x)-(2\beta-\rho)\int_x^t f(t-\tau)\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-x^2})d\tau+$$

$$+\int_x^t f(t-\tau)\exp(-\rho\tau)\left[\frac{\sigma\tau I_1(\sigma\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}\right]d\tau=f(t-x)\exp(-\rho x)+$$

$$+\int_x^t f(t-\tau)\exp(-\rho\tau)\left[\frac{\sigma\tau I_1(\sigma\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}-\sigma I_0(\sigma\sqrt{\tau^2-x^2})\right]d\tau.$$

Thus:

$$\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-x\overline{\mu}(p)\right]\overline{f}(p)\xleftarrow[*]{*}f(t-x)\exp(-\rho x)+$$
$$+\int_x^t f(t-\tau)\exp(-\rho\tau)\left[\frac{\sigma\tau I_1(\sigma\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}-\sigma I_0(\sigma\sqrt{\tau^2-x^2})\right],t>x. \quad (23)$$

Assuming in (23) $\overline{f}(p)=\frac{1}{p}, f(t)=1$, we obtain

$$\frac{1}{p}\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-x\overline{\mu}(p)\right]\xleftarrow[*]{*}\exp(-\rho x)+\int_x^t\exp(-\rho\tau)\left[\frac{\sigma\tau I_1(\sigma\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}-\sigma I_0(\sigma\sqrt{\tau^2-x^2})\right]d\tau,t>x. \quad (24)$$

The discovered originals lead to a number of interesting relations for improper integrals containing Bessel functions. Using the inversion theorem for the Laplace transformation, we can transform (17) into:

$$\frac{1}{\overline{\mu}(p)}\exp\left[-x\overline{\mu}(p)\right]=\int_0^\infty\exp(-\rho t-pt)I_0(\sigma\sqrt{t^2-x^2})\eta(t-x)dt=\int_x^\infty\exp\left[-t(p+\rho)\right]I_0(\sigma\sqrt{t^2-x^2})dt\,. \quad (25)$$

Differentiating both sides of equation (25) with respect to *x*:

$$\exp\left[-x\overline{\mu}(p)\right]=\exp\left[-(\rho+p)x\right]+\sigma x\int_x^\infty\exp\left[-t(\rho+p)\right]\frac{I_1(\sigma\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}}dt. \quad (26)$$

This operation is justified by the uniform convergence of the integral (26). In addition, both sides of the equation (26) are continuous functions with respect to p, therefore, assuming $p \to 0$ we have:

$$\int_x^\infty \exp(-\rho t)\frac{I_1(\sigma\sqrt{t^2-x^2})}{\sqrt{t^2-x^2}}dt = (1/(\sigma x))\left[\exp(-2x\sqrt{\alpha\beta})-\exp(-\rho x)\right]. \tag{27}$$

Assuming (25) $p \to 0$, we obtain:

$$\int_x^\infty \exp(-\rho t)I_0(\sigma\sqrt{t^2-x^2})dt = \frac{1}{2\sqrt{\alpha\beta}}\exp(-2x\sqrt{\alpha\beta}). \tag{28}$$

The next class of images, which is of interest for the thermal shock theory of viscoelastic bodies [2], take the form of:

$$\overline{Q}(x,p) = \frac{1}{p}\exp\left[-xp\sqrt{\frac{p+(\beta_1+\beta_2)}{p+\beta_2}}\right], \tag{29}$$

where $\beta_1 > 0, \beta_2 > 0.$ To clarify the possible form of the original (29), we first study the integral

$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\sqrt{(p+2\alpha)(p+2\beta)}}\exp\left\{\left[-\frac{p}{p+2\beta}x\sqrt{(p+2\alpha)(p+2\beta)}\right]+pt\right\}dp,$ using the methodology

expressed in (11)–(16) for these purposes. We establish that the original $Q(x, t)$ has the following form:

$$Q(x,t) = F(x,t)\eta(t-x), \tag{30}$$

where $F(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{p}\exp\left[-xp\sqrt{\frac{p+(\beta_1+\beta_2)}{p+\beta_2}}+pt\right]dp.$ (31)

The contour integral in (31) has two branch points and is calculated in accordance with the well-known rules of operational calculus [1]. We find that the final result is

$$\frac{1}{p}\exp\left[-xp\sqrt{\frac{p+(\beta_1+\beta_2)}{p+\beta_2}}\right] \xleftarrow[*]{*}$$
$$\xleftarrow[*]{*}\left\{1-\frac{1}{\pi}\int_0^{\beta_1}\frac{1}{y+\beta_2}\exp[-(y+\beta_2)t]\sin\left[x(y+\beta_2)\sqrt{\frac{\beta_1-y}{y}}\right]dy\right\}\eta(t-x). \tag{32}$$

As follows from (32), the original $Q(x, t)$ allows for a jump when passing through the value $t = x$. The magnitude of this jump is

$$\left|\Delta\right| = \lim_{t\to x+0} F(x,t)\eta(t-x) = \lim_{z\to 0} F(x,x+z)\eta(z) = F(x,x) =$$
$$= 1 - \frac{1}{\pi}\int_0^{\beta_1}\frac{1}{y+\beta_2}\exp\left[-(y+\beta_2)x\right]\sin\left[x(y+\beta_2)\sqrt{\frac{\beta_1-y}{y}}\right]dy. \quad (33)$$

When calculating the same value $\left|\Delta\right|$, using the operational approach, we can see that the ratio for the function (30) is as follows:

$$\lim_{p\to\infty}\left[p\overline{Q}(x,p)\exp(px)\right] = F(x,x). \quad (34)$$

Then, in order to prove (34), we begin with the following:

$$\overline{Q}(x,p) = \int_0^\infty \exp(-pt)Q(x,t)dt = \int_x^\infty \exp(-pt)F(x,t)dt = \exp(-px)\int_0^\infty \exp(-pz)F(x,x+z)dz,$$

from which we acquire $\overline{Q}(x,p)\exp(px) = \int_0^\infty \exp(-pz)F(x,x+z)dz.$

Passing to the variable $u = pz$ in the integral, we have

$$p\overline{Q}(x,p)\exp(px) = \int_0^\infty \exp(-u)F(x,x+\frac{u}{p})du.$$

The transition to the limit with respect to (29) and (34) while $p \to \infty$ leads to the following relation

$$\left|\Delta\right| = \lim_{p\to\infty}\left[p\overline{Q}(x,p)\exp(px)\right] = \exp\left[-(\beta_1 x/2)\right]. \quad (35)$$

Finally, an interesting result is derived from the above calculations and (33):

$$1 - \frac{1}{\pi}\int_0^{\beta_1}\frac{1}{y+\beta_2}\exp\left[-(y+\beta_2)x\right]\sin\left[x(y+\beta_2)\sqrt{\frac{\beta_1-y}{y}}\right]dy = \exp\left[-(\beta_1 x/2)\right]. \quad (36)$$

This requires a special explanation (which is supposed to be done in a subsequent publication). From (31), following the rule of differentiating the original $\int_0^t f(\tau)d\tau \xrightarrow[*]{*} (1/p)\overline{f}(p)$ we find another original for the image, which also poses an interest to operational calculus:

$$\exp\left[-xp\sqrt{\frac{p+(\beta_1+\beta_2)}{p+\beta_2}}\right] \xleftarrow[*]{*} \left\{1 - \frac{1}{\pi}\int_0^{\beta_1}\frac{1}{y+\beta_2}\exp\left[-(y+\beta_2)t\right]\sin\left[x(y+\beta_2)\sqrt{\frac{\beta_1-y}{y}}\right]dy\right\}\delta(t-x) +$$
$$\left\{\frac{1}{\pi}\int_0^{\beta_1}\exp\left[-(y+\beta_2)t\right]\sin\left[x(y+\beta_2)\sqrt{\frac{\beta_1-y}{y}}\right]dy\right\}\eta(t-x). \quad (37)$$

Finally, we express the operational solutions to the boundary value problems (2)–(7) in generalized variables:

$\xi = \dfrac{x}{\sqrt{a\tau_r}}, \tau = t/\tau_r, Bi^* = h\sqrt{a\tau_r}, W(\xi,\tau) = \dfrac{T(x,t)-T_0}{T_c - T_0}$ for the cases (4)–(6) and

$W(\xi,\tau) = \dfrac{T(x,t)-T_0}{q_0\sqrt{a\tau_r}/\lambda}$ (for the case of (5)).

We discover:

$$\overline{W}(\xi,p) = \overline{f}(p)\exp\left[-\xi\sqrt{p(p+1)}\right], \quad (38)$$

where $\overline{f}(p) = \dfrac{1}{p}$ in the case of (4), $\overline{f}(p) = \dfrac{p+1}{p^{3/2}}$ in the case of (5), and $\overline{f}(p) = \dfrac{Bi^*\sqrt{p+1}}{p(\sqrt{p} + Bi^*\sqrt{p+1})}$ in case (6).

All originals can be found using the above ratios. It was shown in [3] that the originals of images (38) allow for a transition to new functional constructions that are equivalent to those given above and are very convenient for conducting numerical experiments. This is one of the features of the solutions of hyperbolic transport models for partially bounded domains.

Conclusions

The work has presented the originals of non-standard images, which are part of the operational solutions of a wide class of mathematical models in the theory of transfer (heat, mass, momentum), in network analysis, in the thermal shock theory of viscoelastic bodies, etc. A further development of the problem is the use of cylindrical coordinates for the radial gradients of the physical quantities under examination.

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